S475: A Very Special Theory: An Introduction to Special Relativity
Princeton Splash 2017

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29 April 2017

1 Background

By 1905, it had become clear that there was something seriously wrong with the classical mechanics of Newton that had been a cornerstone of physics since the 17th century, particularly if one desired to hold fast to the belief that the laws of physics should remain unchanged regardless of how fast an observer is travelling. First of all, the laws of electrodynamics, namely, the *Maxwell Equations* were evidently not invariant between different inertial reference frames if one were to use the classical *Galileo transformation*, completely at odds with experimental observations up to that time. In other words, under transformation between a reference frame K to a reference frame K' whose velocity is V along the x-axis relative to frame K, given by:

$$\begin{aligned} x &\to x' = x - Vt \\ y &\to y' = y \\ z &\to z' = z \\ t &\to t' = t \end{aligned} \tag{1.1}$$

the Maxwell Equations:

$$\begin{aligned} \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad & \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad & \vec{B} = \mu_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \epsilon_0 \vec{J} \end{aligned} \tag{1.2}$$

are going to appear much different in K'. Perhaps even more disturbing, the speed of light, which was discovered to be *electromagnetic waves*, was explicitly in the wave equation derived from the Maxwell equation. Should the speed of light not also be considered one of these invariant laws of nature? In addition, it was believed that, as light was a wave, it should require a medium in which to propagate, which physicists of the time called, the *ether*. This ether was not only to be the medium for light, but also provide an "absolute" reference frame for light.

Following this ether theory, one would expect to detect one's motion relative to this ether frame by observing changes in the speed of light. The *Michelson-Morley experiment* of 1887 was devised to do just that, using an interferometer to measure the speed of light at different points in Earth's orbit around the sun (and thus, supposedly, at different velocities relative to the ether). However, no change in the speed of light was observed.

The physicist Hendrik Lorentz had an explanation: It was already known experimentally that the electric field of a point charge, spherically symmetric when the charge is at rest, is contracted when it is moving in the direction of motion. This contraction in the electric potential would thus cause physical objects such as length-measuring devices and time-measuring devices (such as a pendulum) to contract on the macroscopic scale. Thus, lengths would be measured to be longer, and times would be measured to be shorter, thus explaining the apparently constant observed speed of light. He devised a set of coordinate transformations, now known as the *Lorentz transformations*, to explain his theory.

Afterwards it was realized, mainly by Poincare, that under the Lorentz transformation, the Maxwell equations were indeed invariant under change of reference frame! All that was left was for Einstein to realize that, if nature "conspired" for us to measure the speed of light as constant in any reference frame, then as far as we are concerned, the ether is useless for any physical theories we may formulate and thus the speed of light must truly be constant regardless of the velocity of the observer. Thereafter, any old conception of space and time was thrown out.

What was to replace them, first revealed in Einstein's 1905 paper *On the Electrodynamics of Moving Bodies*, was the *special theory of relativity.*

2 Kinematics

2.1 The Postulates

Every single one of the results in the discussion to follow will result solely from the following two assumptions:

1. The laws of nature are invariant under transformation between reference frames.

2. The speed of light is one such invariant law.

Together, these are the *Postulates of Special Relativity*

2.2 The Lorentz Transformation

Now, consider two reference frames, K, and K' whose velocity relative to K is V along the x-axis. Also let the origins of the two reference frames coincide at time $t = t' = 0$. Now let a beam of light, in K, go from space coordinates $\vec{x} = (0, 0, 0)$ at time $t = 0$ (event 1) to $\vec{x} = (0, y, 0)$ at time $t = \frac{y}{c}$ and then to $\vec{x} = (0, 0, 0)$ at time $t = \frac{2y}{c}$ (event 2). In K, the time Δt between events 1 and 2 is $\Delta t = \frac{y}{c}$, and the spatial separation is $\Delta\vec{x} = (0, 0, 0)$ If we didn't require the speed of light to be c in K' as well, then, the time $\Delta t'$ between events 1 and 2 would also be $\Delta t' = \frac{y}{c}$, and an observer in K' would see the light beam going from $\vec{x}' = (0,0,0)$ to $\vec{x}' = (V\frac{y}{c}, y, 0)$ from time $t' = 0$ to time $t' = \frac{y}{c}$, and then from $\vec{x}' = (V\frac{y}{c}, y, 0)$ to $\vec{x}' = (2V\frac{y}{c}, 0, 0)$ from time $t' = \frac{y}{c}$ to time $t' = 2\frac{y}{c}$, where the speed of light would then be $c' = \frac{\sqrt{(V\frac{y}{c})^2 + y^2}}{t} = \sqrt{V^2 + c^2}$. However, under our postulates, this is not allowed.

Thus, suppose that in K, the time difference between events 1 and 2 is some yet unknown $\Delta t'$, such that the spatial separation is $\Delta\vec{x}' = (V\Delta t', 0, 0)$. The order of events is thus, that the light beam goes from space coordinates $\vec{x}' = (0,0,0)$ at time $t' = 0$ (event 1) to $\vec{x}' = (V\Delta t'/2, y, 0)$ at time $t = \Delta t'/2$, and then to $\vec{x} = (V\Delta t', 0, 0)$ at time $t = \Delta t'$ (event 2). We thus have:

$$\begin{aligned} c &= \frac{\sqrt{(V\Delta t'/2)^2 + y^2}}{\Delta t'/2} \\ &= \frac{\sqrt{(V\Delta t'/2)^2 + (c\Delta t/2)^2}}{\Delta t'/2} \\ &= \frac{\sqrt{(V\Delta t'/2)^2 + (c\Delta t/2)^2}}{\Delta t'/2} \\ &= \sqrt{V^2 + c^2\Big(\frac{\Delta t}{\Delta t'}\Big)^2} \end{aligned}$$

or:

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} \tag{2.1}$$

In other words, given K and K' travelling at velocity V along the x-axis relative to K, we have that if two events occur in the *same location* in space in K but are separated by time Δt (the *proper time*), the same two events are separated by a time $\Delta t'$ in K' that has been lengthened by a factor of $\frac{1}{\sqrt{1-(\frac{V}{c})^2}}$. This effect is called *time dilation.*

Now suppose in K, there lies a stick of length Δx at rest, with one end at $\vec{x}_1 = (0,0,0)$ and the other at $\vec{x}_2 = (\Delta x, 0, 0)$. Here, length shall mean the instantaneous simultaneous distance between the ends (the spatial separation of two events occurring at the same time). Now consider the point $\vec{x}' = (0,0,0)$ in K'. This point will fly past the stick in time $\Delta t = \frac{\Delta x}{V}$ in K and in time $\Delta' t = \frac{\Delta x'}{V}$ in K', where $\Delta x = \Delta x'$ might not hold. From time dilation, as the two events of each end of the stick passing by $\vec{x}' = (0,0,0)$ occurs at the same spatial point in K', we have:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \left(\frac{V}{c}\right)^2}}$$

or:

$$\Delta x' = \Delta x\sqrt{1 - \Big(\frac{V}{c}\Big)^2} \tag{2.2}$$

In other words, given K and K' travelling at velocity V along the x-axis relative to K, we have that if two events occur at the *same time* in K but are separated by distance Δx (the *proper length*), the same two events are separated by a distance $\Delta x'$ in K' that has been shortened by a factor of $\sqrt{1-\left(\frac{V}{c}\right)^2}$. This effect is called *length contraction.*

In addition, we have that if, in K, event 1 occurs at time $t = 0$ at the origin (i.e. $\vec{x} = (0,0,0)$) and then event 2 occurs at time $t = \Delta t$, again at the origin, then in K', in K', event 1 occurs at the origin at time $t' = 0$, before event 2 occurs time $t' = \Delta t'$ at $\vec{x} = (-Vt', 0, 0)$, as that is how far the origin of K has traversed in K'. However, $\Delta t' = \frac{\Delta t}{\sqrt{1-(\frac{V}{c})^2}}$. Thus, for $\Delta x = 0$:

$$\Delta x' = -\frac{V\Delta t}{\sqrt{1-\left(\frac{V}{c}\right)^2}} \tag{2.3}$$

This result has been used implicitly in our derivation of time dilation.

In addition, suppose that in K, two pulses of light leave the origin at time $t = 0$. Then, at time $t = \frac{\Delta x}{2c}$, each of them hits a light detector, at rest, located distance $\Delta x/2$ to the right (event 1) and left (event 2) of the origin. Thus, spatial distance between events 1 and 2 is Δx a and the time separation is $\Delta t = 0$. Now, in K', we have that the distance between the two detectors, now in motion, is $l' = \Delta x\sqrt{1-(\frac{V}{c})^2}$. In addition, the right detector is moving towards and the left detector is moving away from the light source at velocity V. Thus, event 1 will occur at $t_1' = \frac{l'/2}{c+V}$ and event 2 will occur at $t_1' = \frac{l'/2}{c-V}$. Thus, for $\Delta t = 0$:

$$\Delta t' = \frac{l'}{1-\left(\frac{V}{c}\right)^2}$$

or

$$\Delta t' = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\frac{\Delta x V}{c^2} \tag{2.4}$$

This reflects a *loss of simultaneity* in special relativity.

As we may simply consider spatial differences relative to the origin, and time differences relative to $t = t' = 0$, we may also drop the Δ's from equations 2.1-4.

Thus far, the effects that we have derived are not yet completely general. For example, in the case of time dilation, 2.1 is only valid if, in K, an event occurs at the origin or if two events occur at the same point in space. However, we may use equations 2.1-4, and their assumptions, as initial conditions to derive a more general form of coordinate transformations between K and K'. Before we do that, let us first define the *Lorentz factor* $\gamma \equiv \frac{1}{1-(\frac{v}{c})^2}$, in order to make our equations more compact. We would thus like to solve the following:

$$x' = Ax + Bt$$
$$t' = Cx + Dt$$

From 2.1, and the assumption that $x = 0$, we have that $D = \gamma$. From 2.3, and the assumption that $x = 0$, we have that $C = -\gamma V$. From 2.2, and the assumption that $t = 0$, we have that $A = \gamma$. From 2.4, and the assumption that $t = 0$, we have that $C = \gamma\frac{V}{c^2}$. Thereby we have replaced the Galileo transformation of equation 1.2 with *Lorentz* transformation:

$$\begin{aligned} x &\to x' = \gamma(x - Vt) \\ y &\to y' = y \\ z &\to z' = z \\ t &\to t' = \gamma\big(t - \frac{Vx}{c^2}\big) \end{aligned} \tag{2.5}$$

The form of the inverse transformation (i.e. from K' to K) is identical up to changing to sign of V.

2.3 Velocity Addition

Taking $x'(x,t)$ and $t'(x,t)$ in K', we have that, for a particle travelling at velocity v in K, such that $x = vt$, we have that, in K', at $t' = \gamma(1 - \frac{Vv}{c^2})t$, the particle will be located at $x' = \gamma(v - V)t$. Thus, if we consider $\frac{\Delta x'}{\Delta t'}$ and take the limit as $\Delta t' \to 0$, we have that the transformation for the particle velocity in K' is given by:

$$v \to v' = \frac{v - V}{1 - \frac{vV}{c^2}} \tag{2.6}$$

This is the form that *velocity addition* takes in special relativity.

Note that this is only for the velocity along the x-axis. For velocities v along the $y-$ or z-axes, we have:

$$v \to v' = \frac{\sqrt{1 - \big(\frac{v}{c}\big)^2}\,v}{1 - \frac{vV}{c^2}} \tag{2.7}$$

It shall also be useful for us to consider how the Lorentz factor transforms. Rather than invoking 2.6 directly, we shall take another approach. Consider the same particle as above in K, such that its proper time τ is scaled by $\gamma(v) = \frac{1}{1-(\frac{v}{c})^2}$ in K (i.e. $t = \gamma(v)\tau$). In K', its proper time τ is scaled by $\gamma(v') = \frac{1}{1-(\frac{v'}{c})^2}$ in K (i.e. $t' = \gamma(v')\tau$). However, at the same time

$$\begin{aligned} t' &= \gamma(V)\big(t - \frac{Vx}{c^2}\big) \\ &= \gamma(V)\big(1 - \frac{Vv}{c^2}\big)t \\ &= \gamma(V)\gamma(v)\big(1 - \frac{Vv}{c^2}\big)\tau \end{aligned}$$

Thus

$$\gamma(v') = \big[\gamma(V)\big(1 - \frac{Vv}{c^2}\big)\big]\gamma(v) \tag{2.8}$$

3 Dynamics

3.1 Momentum

Classically, the momentum $\vec{p}$ of a particle of mass m and velocity $\vec{v}$ is given by:

$$\vec{p} = m\vec{v} \tag{3.1}$$

where mass is thus simply a proportionality constant between momentum and velocity. Such a quantity is desirable for many reasons. First of all, it is conserved in an isolated system free of the influence of external forces. Second of all, given a system of mass m_i with velocities $\vec{v}_i$, we have that in K:

$$\vec{p} = \sum_i \vec{p}_i = \sum_i (m_i \vec{v}_i) = M\vec{v} \tag{3.2}$$

where M is the total mass of the system and $\vec{v}$ is its center of mass. Under Galileo transformation to K', we have:

$$\vec{p'} = \sum_i \vec{p'}_i = \sum_i (m_i \vec{v'}_i) = \sum_i (m_i(\vec{v}_i + \vec{V})) = M(\vec{v} + \vec{V}) \tag{3.3}$$

Thus, momentum allows us to consider a system of particles as a single particle of mass M located and travelling with the center of mass. We would very much like to have a similar quantity for our relativistic theory; however the velocity addition equation 2.6 has complicated this task. We still desire to have a quantity of the form $\vec{p} = m\vec{v}$, however, now, we may have to abandon the invariance of the proportionality constant m when we transform between reference frames.

Consider in K a system consisting of two particles of mass m_1 and m_2 with velocities (along the x-axis) v_1 and v_2 such that:

$$p = m_1 v_1 + m_2 v_2 = 0 \tag{3.4}$$

where K is thereby (without loss of generality) the center of mass frame of the system.

Now consider frame K', where we demand, by analogy 3.3.

$$p' = m_1' v_1' + m_2' v_2' = (m_1' + m_2')V \tag{3.5}$$

However, from 2.6, we have:

$$m_1' \frac{v_1 - V}{1 - \frac{v_1 V}{c^2}} + m_2' \frac{v_2 - V}{1 - \frac{v_2 V}{c^2}}' = (m_1' + m_2')V$$

or, after a bit of algebra and invoking 3.4:

$$\frac{m_1'/m_1}{m_2'/m_2} = \frac{1 - (v_1 V/c^2)}{1 - (v_2 V/c^2)}$$

By use of equation 2.7, we then have:

$$\frac{m_1'/m_1}{m_2'/m_2} = \frac{\gamma(V)}{\gamma(V)} \frac{1 - (v_1 V/c^2)}{1 - (v_2 V/c^2)}$$

We thus have that, without loss of generality, in the case of particle 1:

$$m' = K\gamma(V)\big[1 - (vV/c^2)\big]m$$

Setting both V and v to 0 (the particle is at rest in frame K and K' is stationary with respect to K), we have that $K = 1$. Thus, by comparison with equation 2.6, we see that m transforms exactly the same as γ. Therefore, we have that, m is proportional to *gamma*:

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{3.6}$$

Where m_0 is the constant of proportionality. However, when the particle is at rest ($v = 0$), $m = m_0$, which is thus the mass of the particle in its rest frame, which we shall call the *rest mass* m_0. We have thus found the form of *relativistic mass* which is changed under Lorentz transformation, and also a *relativistic momentum*:

$$\vec{p} = m\vec{v} = \gamma m_0 \vec{v} = \frac{m_0\vec{v}}{\sqrt{1 - \left(\frac{\vec{v}}{c}\right)^2}} \tag{3.7}$$

This definition of momentum, when applied to both particles in our original set-up, satisfies 3.4 and 3.5. Thus, inductively, an arbitrary system of n particles satisfies an analogous form of 3.2 and 3.3:

$$\vec{p} = \sum_i \vec{p}_i = \sum_i (m_i\vec{v}_i) = M\vec{v} \tag{3.8}$$

$$\vec{p'} = \sum_i \vec{p'}_i = \sum_i (m_i\vec{v'}_i) = M'\vec{v'} \tag{3.9}$$

where $\vec{v_i}$ and $\vec{v}'_i$, and $\vec{v}$ and $\vec{v}'$ are pairwise related by the velocity addition equation 2.6, and where:

$$M = \sum_i \frac{m_{0,i}}{\sqrt{1 - \left(\frac{\vec{v_i}}{c}\right)^2}} \tag{3.10}$$

3.2 Energy and Mass

Consider then the mass defined in the last subsection $m = \gamma m_0$. If we multiply this quantity by c^2, we obtain:

$$\begin{aligned} mc^2 &= \gamma m_0 c^2 \\ &= \frac{m_0c^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \\ &= m_0c^2\left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 \cdots\right) \\ &= m_0c^2 + \frac{1}{2}m_0v^2 \cdots \end{aligned} \tag{3.11}$$

where we have taken the binomial expansion. The second term in the expansion of 3.6, we recognize as none other than the classical kinetic energy! Thus, the classical kinetic energy is merely an *approximation* of the full relativistic kinetic energy, minus terms of order higher than 2 in $\frac{v}{c}$. Then, in order for our relativistic theory to reduce to the classical Newtonian theory at low velocities, we must have that our energy E be:

$$E = mc^2 \tag{3.12}$$

where m is the relativistic mass $m = \gamma m_0$. This is the famous *mass-energy equivalence*. One of its immediate consequences is that, for particles of non-zero rest-mass, it requires infinite energy to accelerate it to the speed of light.

We may check that this definition of energy agrees with our classical conceptions of energy. We have that the force is given by:

$$F = \frac{dp}{dt} = \frac{m_0}{\left(1-\left(\frac{v}{c}\right)^2\right)^{3/2}}\frac{dv}{dt} = \gamma^3 m_0 \frac{dv}{dt} \tag{3.13}$$

Thus, we see that only in the instantaneous rest frame of the particle is the classical Newton's second law preserved. We then have that the kinetic energy of the particle T, i.e. the work required to accelerate the particle from rest to velocity v is given by:

$$\begin{aligned} T &= -\int F dx \\ &= -\int \frac{m_0}{\left(1-\left(\frac{v}{c}\right)^2\right)^{3/2}}\frac{dv}{dt}dx \\ &= -\int_0^v \frac{m_0 v}{\left(1-\left(\frac{v}{c}\right)^2\right)^{3/2}}dv \\ &= \frac{m_0c^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} - m_0c^2 \end{aligned} \tag{3.14}$$

To summarize, the kinetic energy of a particle of rest mass m_0 and velocity v is given by the difference between $\gamma m_0 c^2$ and m_0c^2. Thus, $\gamma m_0 c^2$ is implied to be some kind of "total energy," whereas m_0c^2 is some kind of intrinsic, irremovable rest energy. It is also immediate that:

$$E^2 = (m_0c^2)^2 + (\vec{p}c)^2 \tag{3.15}$$

In addition, we have that, in K',

$$\begin{aligned} p' &= \gamma(v')m_0v' \\ &= \gamma(V)\gamma(v)\left[1-\frac{vV}{c^2}\right]m_0\frac{v-V}{1-\frac{vV}{c^2}} \\ &= \gamma(V)\gamma(v)m_0(v-V) \\ &= \gamma\left(p-\frac{VE}{c^2}\right) \end{aligned}$$

We also have:

$$\begin{aligned}
E' &= \gamma(v')m_0c^2 \\
&= \gamma(V)\gamma(v)\big[1 - \frac{vV}{c^2}\big]m_0c^2 \\
&= \gamma(E - Vp)
\end{aligned}$$

Thus, we have the following transformation:

$$\begin{aligned}
p \to p' &= \gamma\big(p - \frac{VE}{c^2}\big) \\
E \to E' &= \gamma(E - Vp)
\end{aligned} \tag{3.16}$$

By analogy with 2.5, we see that energy and momentum obey the same Lorentz transformation as time and space.

4 Unification through Relativity

We have come far from our two innocuous enough postulates. We have uncovered a formal equivalence in space and time, momentum and energy, and the more abstract vector and scalar potentials of a simple relativistic field theory. In this section, we will demonstrate a unification of a slightly different flavor. We will show that the classical magnetic force is fictitious: it is nothing other than the familiar electric force and required consistency with the principles of relativity.

4.1 Charge

We now have to add a third axiom: the electric charge is a Lorentz invariant. This may sound odd at first since it is a very natural idea. Indeed, it certainly feels like a moving electron shouldn't change its charge with motion. In fact, an equivalent phrasing of this postulate is: the laws of physics dictate that the electron has a fundamental charge. This must then be true in all inertial reference frames, and so charge is invariant.

But what is this charge? That is to say, how does one measure it? We know that Maxwell's Laws, rather than Newton's, are relativistic, so we take Gauss's Law to define the charge as follows:

$$\int_{S(t)} \vec{E} \cdot d\vec{A} \equiv Q. \tag{4.1}$$

It is worth emphasizing three things about this definition. First of all, the electric field becomes the primary object in our theory. Second, the velocities of the particles inside S make no difference; thus (thirdly) this definition perfectly fits with our requirement of invariance under boosts. However, because distances change due to length contraction, $d\vec{A}$ is not invariant, which means that $\vec{E}$ cannot be either.

4.2 Transformations of the Field

Say we have a large charged plate such that in the inner region, the electric field is constant. We can use Gauss's Law to determine $|E| = \frac{\sigma}{2}$. Now if we change reference frame to one where the plate is in motion at speed v in the plane of the plate, our Gaussian surface is contracted by a factor of $1/\gamma$. It is quite important that there is only one factor of γ, this is because there is only a length contraction in the direction of the motion. Thus the charge density increases by a factor of γ and $|E| = \frac{\gamma\sigma}{2}$. Now although we have derived this simple result for a simple charge configuration, it holds much more generally. For one, we only appealed to local properties, and two, the field is viewed as its own entity where the nature of the sources that create it is immaterial. Thus we derive that the component perpendicular to the velocity (which boosts us into the primed frame) transforms as

$$E'_\perp = \gamma E_\perp \tag{4.2}$$

and the perpendicular component is unchanged due to the lack of length contraction

$$E'_{||} = E_{||} \tag{4.3}$$

We can use this result to study other important examples of how the field transforms under Lorentz boosts. Perhaps the most obvious one is that of a single point charge.

Consider the field of a proton at rest at the origin where $\vec{E} = \frac{Q}{r^2}\hat{r}$. Thus

$$\begin{aligned} E_x &= \frac{Q}{r^2}\cos\theta = \frac{Qx}{\sqrt{x^2+y^2}^3} \\ E_y &= \frac{Q}{r^2}\sin\theta = \frac{Qy}{\sqrt{x^2+y^2}^3} \end{aligned} \tag{4.4}$$

Now consider the reference frame boosted so that the particle has velocity $\vec{v} = v\hat{x}$ when it passes through the origin. From before, we know that

$$\begin{aligned} E'_x &= E_x = \frac{Qx}{\sqrt{x^2+y^2}^3} \\ E'_y &= \gamma E_y = \gamma\frac{Qy}{\sqrt{x^2+y^2}^3} \end{aligned} \tag{4.5}$$

Now we want to express the field in terms of the coordinates in the primed frame. To simply matters, well choose the time origin so that the particle hits the origin at $t = 0$. In that case, we only need the length contraction formulae: $x = \gamma x'$. Plugging this in, we get

$$\begin{aligned} E'_x &= \frac{Q\gamma x'}{\sqrt{\gamma^2 x'^2 + y'^2}^3} \\ E'_y &= \gamma\frac{Qy'}{\sqrt{\gamma^2 x'^2 + y'^2}^3} \end{aligned} \tag{4.6}$$

The appearance of γ as a coefficient in both components is somewhat surprising! But miraculously, the factors cancel to reproduce the radial dependence of the field. Thus it is natural to think about the magnitude, $\sqrt{E'^2_x + E'^2_y}$. The algebra here is nasty, but ultimately

$$|E| = \frac{Q}{r'^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta')^{3/2}} \tag{4.7}$$

This parameterization is the best suited for generalizing our result for a particle moving in an arbitrary direction. Note that it reduces to the regular expression as $\beta \to 0$. However, when the relativistic corrections are non-negligible, this is a strange field. It is squashed along the direction of motion and thus not spherically symmetric. In fact, it is not conservative! But this should not worry us because this is obviously not an electrostatic field.

4.3 Magnetism

In classical electromagnetism, we say that there exists a magnetic field $\vec{B}$ that produces a force on a moving charge $\vec{F} = Q\vec{v} \times \vec{B}$. We are almost ready to show that this is just a byproduct of relativity. First we need to understand how forces themselves transform as we change reference frames. We know that $\vec{F} = \frac{d\vec{p}}{dt}$, so we only need to study how the momentum and time transform under Lorentz boosts. Again the natural way to split them up is into parallel and perpendicular components. In the object's rest frame K, a force acting upon it produces the usual $\frac{dp_\perp}{dt}$, but when we transform to K', a frame moving with velocity v, there is an effect due to length contraction and time dilation: $t' = \gamma t, x'_{||} = 1/\gamma x_{||}$. In the perpendicular component, the length contraction has no effect, so we only pick up a factor of γ in the denominator. In the parallel direction, although time passes more slowly, lengths are contracted as well. These effects serve to cancel and $F_{||}$ is unchanged. Now we have not been very careful here because this is not the main purpose of the section, but one can go back and use the velocity transform and take appropriate time derivatives carefully to derive this same result:

$$\begin{aligned}\frac{dp'_{||}}{dt'} &= \frac{dp_{||}}{dt} \\ \frac{dp'_\perp}{dt'} &= \frac{1}{\gamma}\frac{dp_\perp}{dt}\end{aligned} \tag{4.8}$$

The final result has a great check! We can see that the transformations of the force exactly cancel the transformations of the E-field. This means that the electric *force* on a moving charge is still qE', where E' is the electric field in that frame - another result that is taken for granted because it seems so natural.

With this result, let's consider the classical magnetic system: a point charge moving parallel to a wire with a current flow. We know that the B-field should point radially from the wire, so there will be a force on the particle. Let's see the real mechanism for the creation of this force.

In the lab frame of the system, the positive charge in the wire (the nuclei of the atoms) are at rest, the electrons in the wire form a current and thus are in motion with speed v_0. The charge moves at speed v. This is the frame in which we want to know the dynamics, but there are many different motions which complicate things. Thus, we will transform to the frame where the test charge is at rest. The has two effects: the electron density and proton density changes. The proton density $\lambda'_+ = \gamma\lambda_0$ due to length contraction. The electron density is a little more complicated because the electrons were not at rest in the first frame, but we can use a result in the earlier section about how the γ factor transforms under multiple shifts of frame. We know that the rest frame charge density must be $-\lambda_0/\gamma_0$, which transforms into the rest frame of the particle as $\lambda'_- = -\frac{\lambda_0}{\gamma_0}\gamma\gamma_0(1-\beta\beta_0)$. So the total charge density in the rest frame of the particle is $\lambda' = \gamma\lambda_0 - \frac{\lambda_0}{\gamma_0}\gamma\gamma_0(1-\beta\beta_0)$. Doing the algebra:

$$\lambda' = \gamma\beta\beta_0\lambda_0 \neq 0 \tag{4.9}$$

Thus, *the wire is charged.* In the rest frame of the particle, there will be a nonzero electric field! This creates a force on the particle $F'_y = -q\frac{2\gamma\beta\beta_0\lambda_0}{r'}$. We can now use the transformation laws to obtain

$$F_y = -q\frac{2\beta\beta_0\lambda_0}{r} \tag{4.10}$$

now this might not look like magnetism, but we can cast it in that form by noting that $-\beta_0\lambda_0 c = I$. Thus, we can rewrite this force in the form

$$F_y = qv_x\frac{2I}{rc^2} \tag{4.11}$$

where we have separated out the magnetic field as $\frac{2I}{rc^2}$. Up to units, this is exactly what is predicted by the classical theory of magnetism.